Second-order Lagrangians admitting a first-order Hamiltonian formalism

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May 10, 2011

Let $p\colon E \to N$ be an arbitrary fibred manifold over a connected n-dimensional manifold N oriented by a volume form $v = dx^1 \wedge \cdots \wedge dx^n$, and let $p^k\colon J^kE \to N$ be the bundle of k-jets of local sections of p, with projections $p^k_l\colon J^kE \to J^lE$ for every $k \geq l$. Every fibred coordinate system (x^j, y^α) on E for the projection p, $1 \leq j \leq n$, $1 \leq \alpha \leq m = \dim E - \dim N$, induces a coordinate system (x^j, y^α_I), on the r-jet bundle, where $I = (i_1, \dots, i_n) \in \mathbb{N}^n$ is an integer multi-index of order $|I| = i_1 + \cdots + i_n \leq r$; namely,

$$y^\alpha_I\left(j^r_x s\right) = \frac{\partial^{|I|}(y^\alpha \circ s)}{\partial(x^1)^{i_1} \cdots \partial(x^n)^{i_n}}(x),$$

where s is a local section of p defined on a neighbourhood of $x \in N$. We use the notations $I = (j) = (0, \dots, 0, \overset{(j}{1}, 0, \dots, 0) \in \mathbb{N}^n$ and $y^\alpha_{(j)} = y^\alpha_j$.

The Legendre form of a second-order Lagrangian density $\Lambda = Lv$ defined on $p\colon E \to N$, where $L \in C^\infty(J^2E)$, is the $V^*(p^1)$-valued p^3-horizontal $(n-1)$-form ω_Λ on J^3E is locally given by (e.g., see [3, 5]),

$$\omega_\Lambda = (-1)^{i-1} L^{i0}_\alpha v_i \otimes dy^\alpha + (-1)^{i-1} L^{i(j)}_\alpha v_i \otimes dy^\alpha_j,$$

where $v_i = dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$, and

$$L^{i(j)}_\alpha = \tfrac{1}{2-\delta_{ij}} \frac{\partial L}{\partial y^\alpha_{(ij)}}, \tag{1}$$

$$L^{i0}_\alpha = \frac{\partial L}{\partial y^\alpha_i} - \tfrac{1}{2-\delta_{ij}} D_j \left(\frac{\partial L}{\partial y^\alpha_{(ij)}} \right), \tag{2}$$

where D_j denotes the "total derivative "with respect to the coordinate x^j, i.e.,

$$D_j = \frac{\partial}{\partial x^j} + \sum_{|I|=0}^{\infty} \sum_{\alpha=1}^{m} y^\alpha_{I+(j)} \frac{\partial}{\partial y^\alpha_I}.$$

The Poincaré-Cartan form attached to Λ is then defined to be the ordinary n-form on J^3E given by (e.g., see [3], [5]),

$$\Theta_\Lambda = (p_2^3)^*\theta^2 \wedge \omega_\Lambda + \Lambda, \tag{3}$$

where θ^2 is the second-order structure form on J^2E locally given in coordinates as follows (cf. [2], [4]):

$$\theta^2 = \left(dy^\alpha - y_i^\alpha dx^i\right) \otimes \frac{\partial}{\partial y^\alpha} + \left(dy_h^\alpha - y_{(hi)}^\alpha dx^i\right) \otimes \frac{\partial}{\partial y_h^\alpha},$$

and the exterior product of $(p_2^3)^*\theta^2$ and the Legendre form, is taken with respect to the pairing induced by duality, $V(p^1) \times_{J^1E} V^*(p^1) \to \mathbb{R}$.

The most outstanding difference with a first-order Lagrangian density is that the Legendre and Poincaré-Cartan forms associated with a second-order Lagrangian density are generally defined on J^3E, thus increasing by one the order of the Lagrangian density Λ.

For certain second-order Lagrangian densities it is well known that the Poincaré-Cartan form is projectable onto J^2E; e.g., see [1]. More precisely, the Poincaré-Cartan form of a second-order Lagrangian projects onto J^2E if and only if the following system of PDEs holds (cf. [1]):

$$\tfrac{1}{2-\delta_{ib}} \frac{\partial^2 L}{\partial y_{ac}^\beta \partial y_{ib}^\alpha} + \tfrac{1}{2-\delta_{ia}} \frac{\partial^2 L}{\partial y_{bc}^\beta \partial y_{ia}^\alpha} + \tfrac{1}{2-\delta_{ic}} \frac{\partial^2 L}{\partial y_{ab}^\beta \partial y_{ic}^\alpha} = 0, \tag{4}$$

for all indices $1 \leq a \leq b \leq c \leq n$, $\alpha, \beta = 1, \ldots, m$.

More surprisingly, there exist second-order Lagrangians for which the associated Poincaré-Cartan form projects not only on J^2E but also on J^1E. Notably, this is the case of the Einstein-Hilbert Lagrange in General Relativity.

In this talk we obtain a characterization of such Lagrangians and we study its Hamiltonian formalism.

References

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